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Sparse Unmixing of Hyperspectral Data based on Robust Linear Mixing Model

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***Abstract*—Recently, sparse unmixing (SU) of hyperspectral data has received particular attention for analyzing remote sensing images. However, most of SU methods are based on the commonly admitted linear mixing model (LMM), which ignores the possible nonlinear effects (i.e. nonlinearity). In this paper, we proposed a new method named robust collaborative sparse regression (RCSR) for hyperspectral unmixing, which is based on the robust LMM (rLMM). The rLMM takes the nonlinearity into consideration, and the nonlinearity is merely treated as outlier, which has the underlying sparse property. The RCSR takes the collaborative sparse property of the abundance and sparsely distributed additive property of the outlier into consideration, which can be formed as a robust joint sparse regression problem. Experiments on synthetic datasets demonstrate that the proposed RCSR is efficient for solving the hyperspectral SU problem compared with other five state-of-the-art algorithms.**

***Index Terms*—Hyperspectral data, outlier, robust collaborative sparse regression, robust LMM, sparse unmixing.**

I. Introduction

Over the last few decades, hyperspectral imaging has been receiving considerable attention in different remote sensing applications such as spectral unmixing, object classification and matching [1]–[4]. Due to insufficient spatial resolution of the imaging sensor and mixing effects of the ground surface, mixed pixels are widespread in hyperspectral images, which lead to difficulties for conventional pixel-level applications. Therefore, spectral unmixing is an essential step for the deep exploitation of hyperspectral image, which decomposes mixed pixels into a collection of pure spectra signatures, called endmembers, and their corresponding proportions in each pixel, called abundances [5].

Sparse unmixing (SU) assumes that the observed image can be formulated as finding the optimal subset of pure spectral signatures from a prior large spectral library, and hence it can be formulated as a linear sparse regression problem. To solve this problem, Bioucas *et al.* proposed sparse unmixing by variable splitting and augmented Lagrangian (SUnSAL) [6]. Besides, some greedy algorithms have been developed for SU, such as the orthogonal matching pursuit (OMP) [7] and subspace matching pursuit SMP [8]. Moreover, Iordache *et al.* proposed collaborative SUnSAL (CLSUnSAL) [9], which improves the unmixing results by adopting the collaborative (also called "multitask" or "simultaneous") sparse regression framework. The above mentioned unmixing algorithms are all based on the commonly admitted linear mixing model (LMM). However, the LMM may be not valid in many situations, for example, when there are multi-scattering effects or intimate interactions, and nonlinear mixing models (NLMMs) provide an alternative to overcome the inherent limitations of LMM [10].

NLMMs have been proposed in the hyperspectral image processing and can be divided into two main classes [10], [11]. The first class of NLMMs consists of physical models based on the nature of the environment. The second class of NLMMs allows for more flexible models for other approximating physics-based models. However, one major drawback of these NLMMs is that they require a specific form of nonlinearity, which makes them limited in practice [12]. Févotte *et al.* proposed the robust LMM (rLMM) [12] to overcome the above mentioned problems, which does not require to specify any analytical form of the nonlinearity. Instead, nonlinearities are merely treated as outliers.

In this paper, to make the SU more flexible for all kinds of HSI unmixing in practice, we proposed a new SU method named robust collaborative sparse regression (RCSR) based on rLMM. The RCSR simultaneously takes the collaborative sparse property of the abundance and sparsely distributed additive property of the outlier into consideration, which can be formed as a robust joint sparse regression problem. The RCSR can be solved by the inexact augmented Lagrangian method (IALM) [13].

II. Methodology

The LMM assumes that the spectral response of a pixel in any given spectral band is a linear combination of all of the endmembers presented in the pixel at the respective spectral band. The LMM can be written as follows:

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}, \tag{1}$$

where $\mathbf{y}$ denotes a $D \times 1$ vector of observed pixel spectra in a hyperspectral image, with D denoting the number of bands, $\mathbf{A} \in \mathbb{R}^{D \times M}$ denotes the endmember signatures, with

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M denoting the number of endmembers, $\mathbf{x} \in \mathbb{R}^{M\times 1}$ is the abundance vector, and $\mathbf{n}$ is the additive noise. For the HSI applications in practice, the abundance sum-to-one constraint (ASC) constraint not always holds true, since signature variability is usually very intensive in an HSI [7]. Therefore, the abundance nonnegativity constraint (ANC) is not taken into consideration. The matrix formulation of LMM is:

$$\mathbf{Y} = \mathbf{AX} + \mathbf{N}, \tag{2}$$

where $\mathbf{Y} \in \mathbf{R}^{D\times B}$ is the collected mixtures matrix, with B denoting the number of pixels, $\mathbf{X} \in \mathbf{R}^{M\times B}$ is the abundance matrix, and $\mathbf{N} \in \mathbf{R}^{D\times B}$ the collected additive noise.

In SU, hyperspectral vectors are approximated by a linear combination of a "small" number of spectral signatures in the library, and the number of columns are equal to the number of pixels, thus the nonzero abundance lines should appear in only a few lines, which implies sparsity along the pixels of an HSI. Since the collaborative (also called "simultaneous" or "multitask") sparse regression approach has shown advantages over the noncollaborative ones, i.e. the mutual coherence has a weaker impact on the unmixing [9], [14]. The CLSUnSAL (also called collaborative hierarchical lasso) imposes sparsity both at group and individual level, which leads to a structured solution as the matrix of fractional abundances contains only a few nonzero lines [9]. It is assumed that the abundance has the underlying collaborative sparse property, which is characterized by the $\ell_{2,1}$ norm, and $\ell_{2,1}$-norm is defined as follows:

$$\|\mathbf{X}\|_{2,1} = \textstyle\sum_{i=1}^{M} \sqrt{\sum_{j=1}^{N} \mathbf{X}_{ij}^2}. \tag{3}$$

Therefore, mathematically, the CLSUnSAL [9] can be written as follows:

$$\min_{\mathbf{X}\geq 0} \|\mathbf{AX} - \mathbf{Y}\|_F + \lambda\|\mathbf{X}\|_{2,1}. \tag{4}$$

However, the CLSUnSAL is based on the LMM, and the LMM may be not valid when there are multi-scattering effects, and NLMMs provide an alternative to overcoming the inherent limitations of the LMM [15]. Févotte *et al.* proposed the rLMM to make the blind unmixing of HSI more flexible to analyze a large variety of remotely sensed scenes, which takes the possible nonlinear effects into consideration, and the nonlinearities are merely treated as outliers [12]. As for the blind unmixing of HSI, the endmember signatures $\mathbf{A}$ and the abundance matrix $\mathbf{X}$ are both unknown. However, for SU of HSI, the endmember signatures $\mathbf{A}$ are selected from a library containing a large number of spectral samples available a priori, and only the abundance matrix $\mathbf{X}$ is unknown. Until now, the rLMM has been not yet used for SU of HSI. Besides, the superiority of rLMM over LMM has been demonstrated in [12].

The rLMM assumes that the spectral response of a pixel in any given spectral band is approximated by a linear combination of all of the endmembers present in the pixel at the respective spectral band and the additive outlier [12].

$$\mathbf{y} = \mathbf{AX} + \mathbf{e} + \mathbf{n}, \tag{5}$$

where $\mathbf{e}$ denotes the outlier term (accounting for nonlinearity). The matrix formulation of rLMM can be written as follows:

$$\mathbf{Y} = \mathbf{AX} + \mathbf{E} + \mathbf{N}, \tag{6}$$

where $\mathbf{Y} \in \mathbf{R}^{D\times B}$ denotes the collected mixtures matrix, with B denoting the number of pixels, $\mathbf{X} \in \mathbf{R}^{M\times B}$ denotes the abundance matrix, $\mathbf{E} \in \mathbf{R}^{D\times B}$ denotes the collected outlier, and $\mathbf{N} \in \mathbf{R}^{D\times B}$ the collected additive noise.

To better pursuit the outlier in rLMM, which has the underlying sparsely distributed additive property. We adopt the $\ell_{2,1}$-norm to impose group sparsity, which has the advantage of rotation invariant compared with the ℓ_1 norm [16]. Therefore, mathematically, the proposed RCSR based on the rLMM can be written as follows:

$$\min_{\mathbf{X}\geq 0,\mathbf{E}} \|\mathbf{AX} + \mathbf{E} - \mathbf{Y}\|_F + \lambda\|\mathbf{X}\|_{2,1} + \alpha\|\mathbf{E}\|_{2,1}, \tag{7}$$

where $\|.\|_F$ represents the matrix Frobenius norm, λ and α are two regularization parameters. Since the rLMM is a generalization of LMM, thus the proposed RCSR is a natural extension of CLSUnSAL with an additional outlier term, which makes the collaborative SU of HSI more robust for outlier.

The optimization problem (7) can be solved by the IALM [13]. By adding the auxiliary matrix $\mathbf{P} \in \mathbf{R}^{M\times N}$, the problem in (7) can be reformulated as follows:

$$\min_{\mathbf{X}\geq 0,\mathbf{E}} \|\mathbf{AP}+\mathbf{E}-\mathbf{Y}\|_F+\lambda\|\mathbf{X}\|_{2,1}+\alpha\|\mathbf{E}\|_{2,1}, \text{ s.t. } \mathbf{X} = \mathbf{P}. \tag{8}$$

Thus, the augmented Lagrangian function can be formed as follows:

$$\begin{aligned}\mathcal{L}(\mathbf{X},\mathbf{E},\mathbf{P}) = \|\mathbf{AP} + \mathbf{E} - \mathbf{Y}\|_F + \lambda\|\mathbf{X}\|_{2,1} + \alpha\|\mathbf{E}\|_{2,1} +& \\ \mathrm{Tr}(\Lambda^T(\mathbf{X} - \mathbf{P})) + \frac{\mu}{2}\|\mathbf{X} - \mathbf{P}\|_F^2,&\end{aligned} \tag{9}$$

then, we apply the alternating minimization scheme to update the seven variables $\mathbf{P}, \mathbf{X}, \mathbf{E}, \Lambda, \mu$, i.e. update one of the five variables with the others fixed.

To update $\mathbf{P}$, we solve

$$\begin{aligned}\mathbf{P}^{k+1} &= \arg\min_{\mathbf{P}} \mathcal{L}(\mathbf{X}^k, \mathbf{E}^k, \mathbf{P}) \\ &= \arg\min_{\mathbf{P}} \|\mathbf{AP} + \mathbf{E}^k - \mathbf{Y}\|_F + \frac{\mu}{2}\|\mathbf{X}^k - \mathbf{P} + \Lambda^k/\mu\|_F^2, \\ &= (2\mathbf{A}^T\mathbf{A} + \mu\mathbf{I})^{-1}[2\mathbf{A}^T(\mathbf{Y} - \mathbf{E}^k) + \mu\mathbf{X}^k + \Lambda^k],\end{aligned} \tag{10}$$

to update $\mathbf{X}$, we solve

$$\begin{aligned}\mathbf{X}^{k+1} &= \arg\min_{\mathbf{X}} \mathcal{L}(\mathbf{X}, \mathbf{E}^k, \mathbf{P}^{k+1}) \\ &= \arg\min_{\mathbf{X}\geq 0} \lambda\|\mathbf{X}\|_{2,1} + \frac{\mu}{2}\|\mathbf{X} - \mathbf{P}^{k+1} + \Lambda^k/\mu\|_F^2,\end{aligned} \tag{11}$$

whose solution is the well-known vect-soft threshold [17], applied independently to each row r of the update variable as follows:

$$\mathbf{X}^{k+1}(r,:) = \max(\text{vect-soft}(\zeta(r,:), \frac{\lambda}{\mu}), 0), \tag{12}$$

where $\zeta = \mathbf{P}^{k+1} - \Lambda^k/\mu$, and vect-soft(b, τ) denotes the row-wise application of the vect-soft-threshold function $g(b,\tau) = b\frac{\max\{\|b\|_2-\tau,0\}}{\max\{\|b\|_2-\tau,0\}+\tau}$. To update $\mathbf{E}$, we solve

$$\begin{aligned}\mathbf{E}^{k+1} &= \arg\min_{\mathbf{E}} \mathcal{L}(\mathbf{X}^{k+1}, \mathbf{E}, \mathbf{P}^{k+1}) \\ &= \arg\min_{\mathbf{E}} \alpha\|\mathbf{E}\|_{2,1} + \|\mathbf{A}\mathbf{P}^{k+1} + \mathbf{E} - \mathbf{Y}\|_F,\end{aligned} \tag{13}$$

which can be also solved by the well-known vect-soft threshold [17]

$$\mathbf{E}^{k+1}(r,:) = \text{vect-soft}(\gamma(r,:), \frac{\alpha}{2}), \tag{14}$$

where $\gamma = \mathbf{Y} - \mathbf{A}\mathbf{P}^{k+1}$. To sum up, the detailed procedure for solving the proposed RCSR is listed in Algorithm 1.

The IALM is a variation of exact augmented Lagrangian method, and its convergence has been studied for at most two blocks (i.e., unknown matrix variables) [18]. For our problem (7), there is no guarantee for the convergence in theory. Furthermore, the IALM is known to generally perform well in reality [18]. In practice, when we choose the parameters appropriately, and it can be observed that the proposed RCSR converges before the maximum iteration is reached.

Algorithm 1: Solving (7) with IALM

```
Input: Y, A;
Output: X̂, Ê;
Initialize X^0, E^0, P^0 Λ^0, λ, α, ρ μ, μ_max = 10^6, k=0;
while convergence criterion is not satisfied do
    P^{k+1} = (2A^T A + μI)^{-1}[2A^T(Y − E^k) + μX^k + Λ^k];
    Compute X^{k+1} by (12);
    Compute E^{k+1} by (14);
    Λ^{k+1} = Λ^k + μ(X^{k+1} − P^{k+1});
    μ = min(ρμ, μ_max);
    k = k + 1;
end
return X̂ = X^{k+1}, Ê = E^{k+1}.
```

III. Experiments

In this section, we carry out simulated experiments to demonstrate the advantage of the proposed RCSR compared with four algorithms based on the LMM, i.e., SUnSAL [6], CLSUnSAL [9], OMP [7] and SMP [8]. To evaluate the performance of different HSI SU algorithms, the signal-to-reconstruction error (SRE) [7] is adopted to measure the power between the signal and error, which is defined as follows:

$$\text{SRE} = 10\log_{10}\left(\frac{\|\mathbf{X}\|_F^2}{\|\mathbf{X}-\hat{\mathbf{X}}\|_F^2}\right), \tag{15}$$

where $\mathbf{X}$ and $\hat{\mathbf{X}}$ are the actual and estimated abundance, respectively. Generally speaking, larger SRE means better hyperspectral sparse unmixing performance.

We use the spectral library randomly selected from the United States Geological Survey (USGS) digital spectral library [1], which has 224 spectral bands uniformly ranging from 0.4 μm to 2.5 μm, and contains 498 spectral signatures of endmembers. We generate the synthetic HSI based on the LMM [7], FM [19], GBM [20] and MGBM [21], and the latter three models are nonlinear unmixing models.

We tune the compared SUnSAL and CLSUnSAL to their best performances by using different regularization parameters: 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1} and 1. The maximum number of iteration and error tolerance of SUnSAL and CLSUnSAL are set to 1000 and 10^{-6}, respectively. For OMP, we set the correct number of endmembers as the input parameter. For SMP, we set the given threshold $\delta = 10^{-3}$. For the proposed RCSR, the performance is tuned to the best by setting λ and α to the following parameters: 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1} and 1. The maximum number of iteration and error tolerance of RCSR is set to the same as these of SUnSAL and CLSUnSAL.

The synthetic HSIs all have 100 $\times$ 100 pixels using endmembers randomly chosen from the USGS library, and all the abundance fractions are generated following the Dirichlet distribution, which satisfy the ANC. The obtained datacubes are then contaminated by Gaussian white noise and correlated noise with different signal-to-noise ratio SNR = $10\log_{10}(\|\mathbf{Y}\|_\mathbf{F}^2/\|\mathbf{N}\|_\mathbf{F}^2)$. Figure 1 shows the performance of SRE as a function of the number of endmember under Gaussian white noise when the SNR is 10 with the LMM, FM, GBM and MGBM, respectively. It can be easily seen from Fig. 1 that the proposed RCSR generally obtains the best SRE, which demonstrates that the RCSR is more robust for outlier than CLSUnSAL. Besides, the performances of most algorithms tend to get worse as the number of endmembers increase, which is due to that the spectral signatures in the spectral library is usually highly correlated.

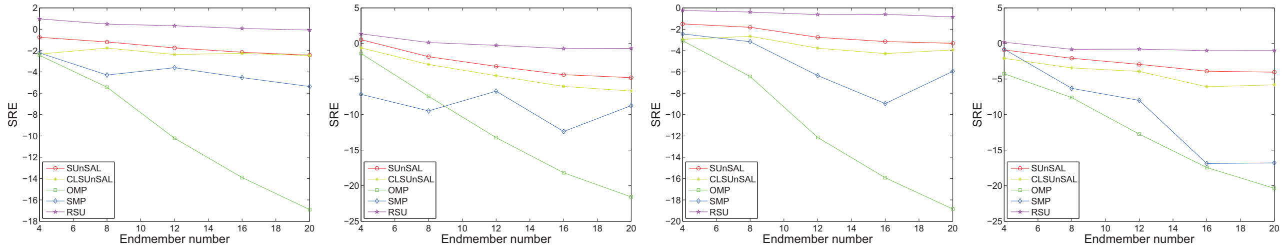
Since it is hard to calibrate the hyperspectral data obtianed from an airborne or spaceborne sensor, the noise and the spectra in real hyperspectral imaging applications are usually low-pass type, which makes the noise highly correlated [7]. Thus, it is very necessary to conduct experiments when the obtained datacubes are contaminated by correlated noise. The synthetic HSI has 100 $\times$ 100 pixels, and all the abundance fractions are generated following the Dirichlet distribution, which satisfy the ASC. The obtained datacubes are then contaminated with correlated noise for different unmixing models, and we generate the correlated noise with the correlated noise function [2]. Figure 2 shows the SU results contaminated by correlated noise when the SNR is 10 with the LMM, FM, GBM and MGBM, respectively. It also can be seen from Fig. 2 that the proposed RCSR generally obtains the best SRE, which demonstrates that the RCSR is more robust for outlier than CLSUnSAL.

IV. Conclusion

In this paper, we propose an RCSR for SU of HSI, which is based on the rLMM. The RCSR take the possible nonlinear effects (i.e. outlier) into consideration, which exploits the

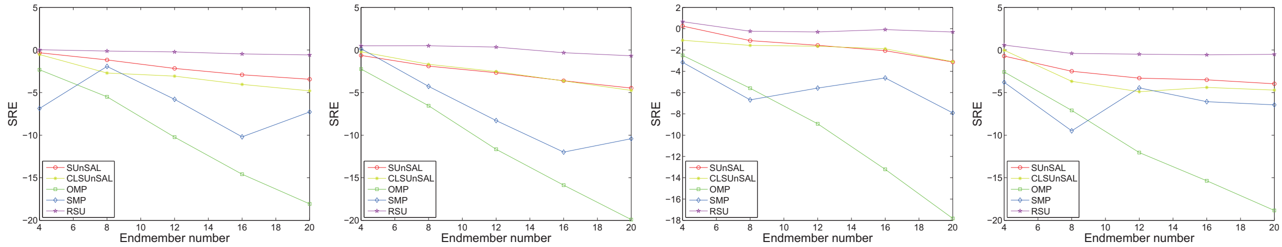
[1] Available at: http://speclab.cr.usgs.gov/spectral-lib.html.

[2] Available at: http://www.mathworks.com/matlabcentral/fileexchange/21156-correlated-Gaussian-noise/content/correlatedGaussianNoise.m.



(a) (b) (c) (d)

Fig. 1: Performance of SRE as a function of endmember number under Gaussian white noise when the SNR is 10. (a) LMM, (b) FM, (c) GBM and (d) MGBM.



(a) (b) (c) (d)

Fig. 2: Performance of SRE as a function of endmember number under correlated noise when the SNR is 10. (a) LMM, (b) FM, (c) GBM and (d) MGBM.

collaborative sparse property of the abundance and sparsely distributed additive property of the outlier. The RCSR can be formed as a robust joint sparse regression problem, which can be solved by the IALM. Experiments on synthetic datasets demonstrate that the proposed RCSR is efficient for solving the hyperspectral unmixing problem compared with the other four state-of-the-art algorithms.

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